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CENTRALLY SYMMETRIC CONVEX BODIES AND THE SPHERICAL RADON TRANSFORM

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1. Introduction

This work comprises part of a study of the relationships between Radon transforms and centrally symmetric convex bodies in d-dimensional Euclidean space $\mathbb{E}^d$, $d \geq 2$. Here we will be concerned with spherical Radon transforms. Radon transforms on higher order Grassmannians are the subject of another paper [13]. The main reason for this division in the program is the different injectivity properties of the various transforms.

The study of centrally symmetric convex bodies is closely connected with the *cosine transform* $T: C_e^\infty(S^{d-1}) \to C_e^\infty(S^{d-1})$ on the infinitely differentiable even functions on the unit sphere S^{d-1} of $\mathbb{E}^d$. For $f \in C_e^\infty(S^{d-1})$, T is defined by

$$(1.1) \qquad (Tf)(u) = \int_{S^{d-1}} |\langle u, v\rangle| f(v)\, \lambda(dv),$$

where $|\langle u, v\rangle|$ denotes the absolute value of the scalar product of u, $v \in S^{d-1}$, and λ is the spherical $(d-1)$-dimensional Lebesgue measure on S^{d-1}. The total surface area measure, $\lambda(S^{d-1})$, of S^{d-1} will be denoted by ω_{d-1}. Background information on the geometric aspects of this transform can be found in the survey article by Schneider and Weil [31]. One of our primary objectives in this work is to investigate and use the relationship between the cosine transform and the *Radon transform* $R: C_e^\infty(S^{d-1}) \to C_e^\infty(S^{d-1})$ defined by

$$(1.2) \qquad (Rf)(u) = \frac{1}{\omega_{d-2}} \int_{S_u^{d-2}} f(v)\, \lambda_u(dv), \qquad u \in S^{d-1},$$

where λ_u denotes the $(d-2)$-dimensional spherical Lebesgue measure on the great subsphere S_u^{d-2} comprising the elements of S^{d-1} orthogonal to u.

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Both these transforms have been extensively studied, but their connections have only occasionally been investigated (see Blaschke [3], Petty [23], and Schneider [27], [29]). The connection between R and T will enable us to apply properties of the Radon transform to solve some geometric problems. In the reverse direction we will use a result of Schneider and Weil [30], about projections of convex bodies, to establish some new support properties for R.

The second section provides a comparison of the properties of the spherical Radon transform and the cosine transform and will show the relevance of these transforms to the study of centrally symmetric convex bodies. In §3 we will use Radon transform techniques to give a short approach to Berg's [2] solution of the Christoffel problem when restricted to the case of centrally symmetric bodies. In §4 we will find smoothness conditions which guarantee that a body is a generalized zonoid. This uses techniques of Schneider [26] and enables us to give improved estimates for the degree of the generating distribution of centrally of centrally symmetric bodies. The paper concludes with our discussion of zonal characterizations of zonoids and support properties of Radon transforms. Here we use the support properties of the Radon transform to find a zonal characterization of zonoids in the case of even dimensions; the same result was recently obtained by Panina [22], using completely different techniques.

2. Centrally symmetric bodies and spherical transforms

Both the transforms R and T described in §1 satisfy certain duality conditions. For $f,\ g \in C_e^\infty(S^{d-1})$

$$\int_{S^{d-1}} f(u)(Rg)(u)\,\lambda(du) = \int_{S^{d-1}} (Rf)(u)g(u)\,\lambda(du) \tag{2.1}$$

and

$$\int_{S^{d-1}} f(u)(Tg)(u)\,\lambda(du) = \int_{S^{d-1}} (Tf)(u)g(u)\,\lambda(du). \tag{2.2}$$

Equality (2.1) can be found in standard texts such as Helgason [15], [16], whereas (2.2) is a simple consequence of Fubini's theorem. We shall denote by $\mathscr{D}_e(S^{d-1})$ the space of even distributions on S^{d-1}; this is the dual of the space $C_e^\infty(S^{d-1})$ endowed with the topology of uniform convergence of all derivatives. Since R and T are continuous linear mappings on $C_e^\infty(S^{d-1})$, their transposes R^* and T^* are linear mappings from $\mathscr{D}_e(S^{d-1})$ to itself. (2.1) and (2.2) show that R^* and T^* are extensions of R and T, so we may think of R and T as mappings of $\mathscr{D}_e(S^{d-1})$ to itself.

It is well known that R and T are injective when thought of as mappings of $C_e^\infty(S^{d-1})$ to itself. For $d = 3$, the injectivity of R is due to Minkowski [20] and is also proved in Funk [12] and Bonnesen and Fenchel [7]. In the case $d \geq 3$, Helgason [14] gives inversion formulas for R; proofs of the injectivity of R can also be found in Petty [23] and Schneider [27]. Blaschke [3] gives proofs of the injectivity of T in case $d = 3$. The general case is a consequence of a result of Alexandrov [1]; he established the injectivity of T on the space of measures. If $\mu \in \mathscr{M}_e(S^{d-1})$, the space of even signed measures on S^{d-1}, with

$$\int_{S^{d-1}} |\langle u, v\rangle| \, \mu(dv) = 0 \quad \text{for all } u \in S^{d-1}, \tag{2.3}$$

then $\mu \equiv 0$. Further proofs can be found in Choquet [8], [9], Matheron [18], [19], Petty [23], Rickert [24], [25], and, in a more general setting, Schneider [28]. In fact R and T are both continuous bijections of $C_e^\infty(S^{d-1})$ to itself. This result, for R, can be found in Helgason [16], for example; for T it is a consequence of a result of Schneider [26]. Now $C_e^\infty(S^{d-1})$ is complete and metrizable, and so the Open Mapping Theorem shows that R^{-1} and T^{-1} are continuous mappings of $C_e^\infty(S^{d-1})$ to itself. From this we deduce that R and T are continuous bijections of $\mathscr{D}_e(S^{d-1})$ to itself, when $\mathscr{D}_e(S^{d-1})$ is given the strong topology. These injectivity properties of R are the ones to which we alluded in the introduction.

The relationship between R and T involves the Laplacian Δ on S^{d-1}. In order to see this we recall from Berg [2] that if K is a convex body with support function $h(K;\cdot)$ and first surface area measure $S_1(K;\cdot)$, then

$$((d-1)^{-1}\Delta + 1)h(K;\cdot) = S_1(K;\cdot) \tag{2.4}$$

as distributions; the reader is referred to Bonnesen and Fenchel [7] or Leichtweiss [17] for information on standard notions in the study of convex bodies. For convenience we will put $\Box = (\Delta + d - 1)/2\omega_{d-2}$. This enables us to prove the following proposition which, in case $d = 3$, was observed by Blaschke [3].

Proposition 2.1. *If R is the spherical Radon transform and T is the cosine transform, then*

$$\Box T = R. \tag{2.5}$$

Proof. The result follows immediately from the observation that if K is the line segment with endpoints $\pm u \in S^{d-1}$, then (1.2) and (2.4) give

$$(2.6) \qquad (\Box(|\langle u, \cdot\rangle|))(f) = (Rf)(u), \qquad f \in C_e^{\infty}(S^{d-1}). \quad \text{q.e.d.}$$

The intertwining properties of R and (2.5) allow us to deduce that

$$T^{-1} = R^{-1}\Box = \Box R^{-1}.$$

There are inversion formulas for R which, in the case of even dimensions, involve polynomial expressions in Δ. Helgason [14] shows that, for even d, there is a polynomial p_d of degree $d/2$ such that for $f \in C_e^{\infty}(S^{d-1})$

$$(2.7) \qquad f = p_d(\Delta)R^2 f.$$

In the case of odd dimensions, some formulations of the inversion formulas involve $\Delta^{1/2}$. We will investigate these more thoroughly in §4. For the present, we note that for even values of d the inversion formula (2.7) gives

$$(2.8) \quad T^{-1}f = c\prod_{i=0}^{d/2}(\Delta - (2i-1)(d-2i-1))Tf, \qquad f \in C_e^{\infty}(S^{d-1}),$$

where c is a constant dependent only on d.

The above observations play an important role in connection with integral representations for support functions of centrally symmetric convex bodies. In order to see this, we recall that a *zonotope* is a vector sum of line segments. The support function of a zonotope K, with center at the origin, is therefore of the form

$$(2.9) \qquad h(K; u) = \sum_{i=1}^{n} |\langle u, v_i\rangle| \rho_K(v_i), \qquad u \in S^{d-1},$$

where, for $i = 1, \cdots, n$, the v_i are unit vectors, and the $\rho_K(v_i)$ are positive numbers. We will denote by $\mathscr{K}_0$ the class of *centered* convex bodies, that is, those centrally symmetric bodies with center at the origin. *Zonoids* are limits of zonotopes and, for these bodies, (2.9) is replaced by the integral representation

$$(2.10) \qquad h(K; u) = \int_{S^{d-1}} |\langle u, v\rangle| \, \rho_K(dv), \qquad u \in S^{d-1},$$

where ρ_K is a positive even measure on S^{d-1}, called the *generating measure* of K. Alexandrov's uniqueness result (2.3) shows that the generating measure is uniquely determined by the body K.

Zonotopes are highly symmetric and are characterized by the fact that all their two-dimensional faces are centrally symmetric. It is therefore not surprising to learn that zonoids also exhibit a very high degree of symmetry and so are not dense among centrally symmetric bodies. There are convex bodies K whose support functions have the integral representation (2.10) with a signed measure ρ_K; these bodies, together with their translates are called *generalized zonoids*. Specific examples of generalized zonoids which are not zonoids are given in Schneider [28]. The existence of these bodies demonstrates the lack of denseness of zonoids in $\mathscr{K}_0$, although this is also a consequence of an earlier result of Shephard [33]. The generalized zonoids are dense among centrally symmetric convex bodies (see Schneider [26]). This denseness led Weil [36] to show that each centrally symmetric convex bodies has a *generating distribution.* In fact the surjectivity of $T\colon \mathscr{D}_e(S^{d-1}) \to \mathscr{D}_e(S^{d-1})$ allows us to define the generating distribution ρ_K of $K \in \mathscr{K}_0$ to be $T^{-1}(h(K;\cdot))$. So ρ_K satisfies

$$(2.11) \qquad \rho_K(f) = \int_{S^{d-1}} h(K;u)(T^{-1}f)(u)\,\lambda(du), \qquad f \in C_e^\infty(S^{d-1}).$$

Clearly the domain of ρ_K can be extended to include any function f of the form

$$f(\cdot) = \int_{S^{d-1}} |\langle \cdot, v\rangle| \rho_f(dv)$$

for some $\rho_f \in \mathscr{M}_e(S^{d-1})$. For such a function f we have

$$\rho_K(f) = \int_{S^{d-1}} h(K;u)\,\rho_f(du);$$

in particular, as Weil [36] showed, we have the following analogue of (2.10) for arbitrary $K \in \mathscr{K}_0$:

$$(2.12) \qquad \rho_K(|\langle u, \cdot\rangle|) = h(K;u).$$

Alexandrov's result (2.3) shows that this extension of ρ_K is well defined. So the generating distribution ρ_K can be extended to a much wider class of functions. We will investigate this extension in more detail in §4. For the moment, we will just note that we now have a heirarchy of centrally symmetric bodies corresponding to the nature of the generating distribution. Zonotopes are those bodies for which the generating distribution is an atomic measure, zonoids are those for which it is a positive measure, and generalized zonoids are those for which it is a signed measure. In fact this heirarchy can also be seen from a slightly different point of view. If $K \in \mathscr{K}_0$, then (2.4), (2.6), and (2.12) show that

$$(2.13) \qquad (d-1)S_1(K;\cdot) = 2\omega_{d-2}\Box h(K;\cdot) = 2\omega_{d-2}R\rho_K.$$

So we see from (2.13) that zonoids (respectively, generalized zonoids) are the bodies whose first surface area measures are Radon transforms of positive (respectively, signed) measures. Results of the form (2.13) appeared in Weil [35].

3. The Christoffel problem for centrally symmetric bodies

The Christoffel problem asks for conditions on a measure μ on S^{d-1} which guarantee that it is the first surface area measure of a convex body. This problem was solved independently by Berg [2] and Firey [10], [11]. Here we want to give a short approach to Berg's solution in the case of centrally symmetric bodies. So we seek conditions on $\mu \in \mathcal{M}_e(S^{d-1})$ which ensure that there is a $K \in \mathcal{K}_0$ with $\mu = S_1(K;\cdot)$.

We notice that if it were possible to find a function $f_d \in L^1(S^{d-1})$ with

$$Rf_d = |\langle u, \cdot\rangle| \tag{3.1}$$

for fixed $u \in S^{d-1}$, then f_d must be rotationally symmetric in the sense that there is a $g_d \in L^1([0,1])$ with $f_d = g_d(|\langle u, \cdot\rangle|)$. Thus, formally, (2.12), (2.13), and (3.1) would imply that

$$2\omega_{d-2}Rh(K;\cdot) = 2\omega_{d-2}R\rho_K(|\langle\cdot,\cdot\rangle|) = (d-1)R\int_{S^{d-1}} g_d(|\langle\cdot, v\rangle|)\,S_1(K;dv),$$

and therefore, because of the injectivity of R, we could deduce that

$$2\omega_{d-2}h(K;\cdot) = (d-1)\int_{S^{d-1}} g_d(|\langle\cdot, v\rangle|)\,S_1(K;dv).$$

This motivates our proof of the following result of Berg [2].

Theorem 3.1. *The measures* $\mu \in \mathcal{M}_e(S^{d-1})$ *for which*

$$\int_{S^{d-1}} g_d(|\langle u, v\rangle|)\,\mu(dv)$$

is a convex function (of u*) are precisely those which are first order surface area measures of centrally symmetric convex bodies.*

Proof. We assume that $\mu \in \mathcal{M}_e(S^{d-1})$ and that there is a $K \in \mathcal{K}_0$ with

$$\int_{S^{d-1}} g_d(|\langle u, v\rangle|)\,\mu(dv) = \frac{2\omega_{d-2}}{d-1}h(K;u) \tag{3.2}$$

for all $u \in S^{d-1}$, and we aim to show that $\mu = S_1(K;\cdot)$. To this end let $g \in C_e^\infty(S^{d-1})$; then there is an $f \in C_e^\infty(S^{d-1})$ such that $g = Tf$. It follows from (1.1), (2.1), and (3.1) that

$$\begin{aligned}\int_{S^{d-1}} g(u)\,\mu(du) &= \int_{S^{d-1}} (Tf)(u)\,\mu(u)\\ &= \int_{S^{d-1}}\int_{S^{d-1}} |\langle u, v\rangle| f(v)\,\lambda(dv)\,\mu(du)\\ &= \int_{S^{d-1}}\int_{S^{d-1}} (Rg_d(|\langle u, \cdot\rangle|))(v) f(v)\,\lambda(dv)\,\mu(du)\\ &= \int_{S^{d-1}}\int_{S^{d-1}} g_d(|\langle u, v\rangle|)(Rf)(v)\,\lambda(dv)\,\mu(du).\end{aligned}$$

Now we can use the fact that g_d is an L^1 function to apply Fubini's theorem which, together with (2.5) and (3.2), gives

$$\begin{aligned}(d-1)\int_{S^{d-1}} g(u)\,\mu(du) &= 2\omega_{d-2}\int_{S^{d-1}} h(K; v)(aRf)(v)\,\lambda(dv)\\ &= 2\omega_{d-2}\int_{S^{d-1}} h(K; v)(\Box Tf)(v)\,\lambda(dv)\\ &= 2\omega_{d-2}(\Box h(K; \cdot))(g)\\ &= (d-1)\int_{S^{d-1}} g(v)\,S_1(K; dv).\end{aligned}$$

Since g was an arbitrary member of $C_e^\infty(S^{d-1})$, it follows that $\mu = S_1(K; \cdot)$ as required. q.e.d.

In order to find the required function f_d, or g_d, we investigate the action of R on rotationally symmetric functions. It is clear that if f is such a function, then so is Rf and therefore there is a function $\widetilde{R}\colon L^1([0, 1]) \to L^1([0, 1])$ associated with R.

Proposition 3.2. *For $g \in L^1([0, 1])$ and $r \in [0, 1]$ we have*

$$(\widetilde{R}g)(r) = 2\left(\frac{\omega_{d-3}}{\omega_{d-2}}\right)(1-r^2)^{\frac{-(d+3)}{2}}\int_0^{\sqrt{1-r^2}} g(s)(1-r^2-s^2)^{\frac{(d-4)}{2}}\,ds, \tag{3.3}$$

for $d \geq 3$, and

$$(\widetilde{R}g)(r) = g((1-r^2)^{1/2}) \tag{3.4}$$

for $d = 2$.

Proof. We recall that if n is the north pole of S^{d-1} $(d \geq 3)$ and S^{d-1} is parametrized by cylindrical coordinates $u(t, v)$, where $t \in [-1, 1]$ and $v \in S_n^{d-2}$, then

$$\lambda(du) = (1-t^2)^{(d-3)/2}\,dt\lambda_n(dv).$$

So

$$(\widetilde{R}g)(r) = 2\left(\frac{\omega_{d-3}}{\omega_{d-2}}\right)\int_0^{(1-r^2)^{1/2}} g(s)\left(1-\frac{s^2}{1-r^2}\right)^{(d-4)/2}(1-r^2)^{-1/2}\,ds,$$

which gives the required result in case $d \geq 3$. The case $d = 2$ is trivial. q.e.d.

From (3.1) and (3.4) it follows that

$$g_2(r) = (1-r^2)^{1/2}.$$

For $d \geq 3$, (3.3) shows that we require a solution g_d of the integral equation

$$r(1-r^2)^{(d-3)/2} = 2\left(\frac{\omega_{d-3}}{\omega_{d-2}}\right)\int_0^{(1-r^2)^{1/2}} g_d(s)(1-r^2-s^2)^{(d-4)/2}\,ds.$$

A simple change of variable yields that this is equivalent to the integral equation

$$(1-r^2)^{1/2} = 2\left(\frac{\omega_{d-3}}{\omega_{d-2}}\right)\int_0^1 g_d(sr)(1-s^2)^{(d-4)/2}\,ds.$$

This is a Fredholm integral equation which, in case $d = 3$, has a singular kernel. Integration by parts gives that

$$2\left(\frac{\omega_{d-1}}{\omega_d}\right)\int_0^{(1-r^2)^{1/2}} [sg_d'(s) + (d-1)g_d(s)](1-r^2-s^2)^{(d-2)/2}\,ds$$
$$= (d-1)r(1-r^2)^{(d-1)/2},$$

and so we obtain the following recurrence relation, which is valid for $d \geq 3$:

$$g_{d+2}(r) = \frac{r}{d-1}g_d'(r) + g_d(r).$$

Thus we are able to give a specific formulation of all the g_d, which are essentially the even part of Berg's [2] functions. In order to facilitate comparisons with his functions, we list the results for small dimensions:

$$g_3(r) = 1 + \frac{r}{2}\log_e\frac{1-r}{1+r},$$
$$g_4(r) = (1-2r^2)(1-r^2)^{-1/2},$$
$$g_5(r) = \frac{1}{2}(2-3r^2)(1-r^2)^{-1} + \frac{3r}{4}\log_e\frac{1-r}{1+r},$$
$$g_6(r) = \frac{1}{3}(8r^4-12r^2+3)(1-r^2)^{-3/2}.$$

4. Smooth centrally symmetric bodies

We mentioned in §2 that the generalized zonoids are dense in the class of centrally symmetric bodies. This follows from a result of Schneider [26] which states that for $h(K; \cdot) \in C_e^k(S^{d-1})$, the space of k-times continuously differentiable even functions on S^{d-1}, K is a generalized zonoid if $k = d + 2$ when d is even and $k = d + 3$ when d is odd. In fact his result shows that for such a body K, the generating distribution ρ_K is not only a measure but a continuous function. In this section we will use his techniques [26] to find conditions which force ρ_K to be an L^2 function. We recall that $f \in L^2(S^{d-1})$ if and only if its spherical harmonic expansion $\sum_{n=0}^{\infty} f_n$ satisfies

$$\sum_{n=0}^{\infty} \|f_n\|_2^2 < \infty,$$

where each f_n is an eigenvector of the Laplacian with

$$\Delta f_n = -n(n + d - 2) f_n; \tag{4.1}$$

see Müller [21] for more details. We will work with the spaces $L_s^2(S^{d-1})$, $s \geq 0$, of those functions f for which the spherical harmonic expansion satisfies

$$\|f\|_{L_s}^2 = \sum_{n=0}^{\infty} (1 + n^2)^s \|f_n\|_2^2 < \infty.$$

These are precisely the functions f on S^{d-1} with derivations up to order s in $L^2(S^{d-1})$, and so the $L_s^2(S^{d-1})$, $s \geq 0$, are the Sobolev spaces on S^{d-1}. Strichartz [34] analyzes the action of the Radon transform on these spaces by using the fact that the spherical harmonics are also eigenvalues of R to show that

$$R\colon L_s^2(S^{d-1}) \to L_{s+(d-2)/2}^2(S^{d-1})$$

is a bijection. He also proves that there is a constant b such that

$$b^{-1} \|Rf\|_{L_{s+(d-2)/2}} \leq \|f\|_{L_s} \leq b \|Rf\|_{L_{s+(d-2)/2}}$$

for all $f \in L_s^2(S^{d-1})$. From (2.5) and (4.1) it follows that

$$T\colon L_s^2(S^{d-1}) \to L_{s+(d+2)/2}^2(S^{d-1}) \tag{4.2}$$

is a bijection and that there is a constant c such that

(4.3) $$c^{-1}\|Tf\|_{L_{s+(d+2)/2}} \le \|f\|_{L_s} \le c\|Tf\|_{L_{s+(d+2)/2}}.$$

Now by (2.12) we have $\rho_K = T^{-1}h(K;\cdot)$ for $K \in \mathscr{K}_0$. So any $K \in \mathscr{K}_0$ for which $h(K;\cdot) \in L^2_{(d+2)/2}(S^{d-1})$ must be a generalized zonoid having a generating measure which is an L^2 function. But if $f \in C^k_e(S^{d-1})$ has spherical harmonic expansion $\sum_{n=0}^{\infty} f_n$, then, by (4.1), $\Delta^{k/2}f$ has spherical harmonic expansion

(4.4) $$\sum_{n=0}^{\infty}(-1)^{k/2}n^{k/2}(n+d-2)^{k/2}f_n.$$

Since this is a continuous function, we deduce that $f \in L^2_k(S^{d-1})$. So $C^k_e(S^{d-1}) \subset L^2_k(S^{d-1})$ for k even. An analogous argument shows that $C^k_e(S^{d-1}) \subset L^2_{k-1}(S^{d-1})$ for k odd. Hence we have the following result.

Theorem 4.1. *If $K \in \mathscr{K}_0$ with $h(K;\cdot) \in C^k_e(S^{d-1})$, then K is a generalized zonoid for each of the following cases:*

(i) $k = (d+4)/2$ *in case* $d \equiv 0 \bmod (4)$,
(ii) $k = (d+3)/2$ *in case* $d \equiv 1 \bmod (4)$,
(iii) $k = (d+2)/2$ *in case* $d \equiv 2 \bmod (4)$, *and*
(iv) $k = (d+5)/2$ *in case* $d \equiv 3 \bmod (4)$.

We recall that since S^{d-1} is compact, every distribution on S^{d-1} is a sum of derivatives of finite orders of measures of S^{d-1}; see Schwartz [32]. The maximum order of these derivatives is called the *degree* of the distribution. As explained by Weil [36], we can use the results above to find a bound for the degree of the generating distribution of an arbitrary body $K \in \mathscr{K}_0$.

Theorem 4.2. *If $K \in \mathscr{K}_0$, then ρ_K has degree at most $(d+5)/2$.*

Proof. We let k be an even number with $(d+2)/2 \le k \le (d+5)/2$. It suffices to show that there is a constant α such that

(4.5) $$|\rho_K(f)| \le \alpha\|\Delta^{k/2}f\|_\infty$$

for all $f \in C^k_e(S^{d-1})$. We know from the above discussion that $f \in L^2_k(S^{d-1})$, and so there is a $g \in L^2(S^{d-1})$ with $f = Tg$, in consequence of (4.2). Thus by (2.11) and (4.3), there is a constant c_1 such that

$$|\rho_K(f)| \le \|h(K;\cdot)\|_2\|g\|_2 \le c_1\|h(K;\cdot)\|_2\|f\|_{L_{(d+2)/2}}.$$

But (4.4) implies that there are constants c_2 and c_3 such that

$$\|f\|^2_{L_{(d+2)/2}} = \sum_{n=0}^{\infty}(1+n^2)^{(d+2)/2}\|f_n\|_2^2 \le c_2 \sum_{n=0}^{\infty} n^{2k}\|f_n\|_2^2$$
$$\le c_3 \sum_{n=0}^{\infty} \|\Delta^{k/2} f_n\|_2^2 = c_3\|\Delta^{k/2} f\|_2^2 .$$

Inequality (4.5) now follows from the fact that S^{d-1} is compact, and so the L^2 norm is dominated by the supremum norm.

It would be interesting to know if there is an upper bound for the degree of ρ_K independent of the dimension d.

5. Zonal characterization of zonoids

Our aim in this section is to give a partial verification of a conjecture of Weil [37]. To be precise, we will show that, in even dimensions, there is a zonal characterization of zonoids. This result has also been established by Panina [22] using completely different techniques. We shall obtain the result as a consequence of certain support properties of Radon transforms and prove that such support properties are not valid in the case of odd dimensions.

First we will give some of the background to the problem. Blaschke [4] (see also Blaschke and Reidemeister [5]) and Bolker [6] asked for a local characterization of zonoids. Such a characterization would imply that any $K \in \mathscr{K}_0$ with the property that for each $u \in S^{d-1}$ there is a neighborhood $U_u \subset S^{d-1}$ of u and a zonoid Z_u with

$$h(K;\cdot) = h(Z_u;\cdot) \quad \text{on } U_u \tag{5.1}$$

must itself be a zonoid. Weil [37] showed this is false, constructing counterexamples for all dimensions $d \ge 3$. He went on to ask whether a zonal characterization is possible, that is, whether (5.1) characterizes zonoids, if, instead of a neighborhood U_u of u, a neighborhood E_u of the equator $u^\perp$ is considered. Such a neighborhood is called an *equatorial zone*.

Theorem 5.1. *Assume d is even. If $K \in \mathscr{K}_0$ has the property that for every equator $u^\perp$, $u \in S^{d-1}$, there is an equatorial zone $E_u \subset S^{d-1}$ and a zonoid $Z_u = Z(E_u)$ such that*

$$h(K;\cdot) = h(Z_u;\cdot) \quad \textit{on } E_u, \tag{5.2}$$

then K is a zonoid.

Proof. It suffices to show that the generating distribution ρ_K is a positive distribution (see, for example, Schwartz [32]). So we choose $g \in C_e^\infty(S^{d-1})$ with $g \geq 0$ and aim to show that $\rho_K(g) \geq 0$. Let U_u be the (symmetric) *cap* comprising all unit vectors orthogonal to some vector in E_u. These caps cover S^{d-1}, and a partition of unity argument shows that we can assume that g is supported on U_u for some $u \in S^{d-1}$. We note from (2.5) and (2.7) that there is a polynomial p_d such that

$$T^{-1}g = \Box p_d(\Delta)Rg,$$

which is equivalent to (2.8). Now Rg is clearly supported on E_u and so the same must be true of $T^{-1}g$. Therefore (2.11) and (5.2) give

$$\begin{aligned}\rho_K(g) &= \int_{S^{d-1}} h(K; v)(T^{-1}g)(v)\,\lambda(dv) = \int_{E_u} h(K; v)(T^{-1}g)(v)\,\lambda(dv)\\ &= \int_{E_u} h(Z_u; v)(T^{-1}g)(v)\,\lambda(dv) = \rho_{Z_u}(g) \geq 0.\end{aligned}$$

Hence ρ_K is a positive distribution, and the proof is complete. q.e.d.

We note that in the above proof, the important step was the following support property of R, which is a consequence of (2.7).

Proposition 5.2. *Let d be even and let $g \in C_e^\infty(S^{d-1})$. If Rg is supported on a (symmetric) cap C, then the support of g is contained in the orthogonal zone $C^\perp$. Equivalently, if $Rg \equiv 0$ on $C^\perp$ for some cap C, then $g \equiv 0$ on C.*

We conclude this work by showing that this result is not true in odd dimensions. To see this we first recall a result of Schneider and Weil [30].

Proposition 5.3. *Let d be odd and let $f \in C_e^\infty(S^{d-1})$. If $f \equiv 0$ on a cap C and $Tf \equiv 0$ on $C^\perp$, then $f \equiv 0$ on S^{d-1}.*

If d is odd and $Rg \equiv 0$ on $C^\perp$, then (2.5) yields that the same must be true of $T\Box g$. If we could conclude that $g \equiv 0$ *on* C then, of course, $\Box g \equiv 0$ on C. So Proposition 5.3 shows that $\Box g \equiv 0$ on S^{d-1}, which implies $g \equiv 0$ on S^{d-1} (see Berg [2]). But this is impossible since the surjectivity of R: $C_e^\infty(S^{d-1}) \to C_e^\infty(S^{d-1})$ shows that there are nontrivial functions whose Radon transforms vanish on an equatorial zone.

Schneider and Weil [30] proved, by means of a counterexample, that Proposition 5.3 is false in even dimensions. We now see that the latter observation is also a consequence of Proposition 5.2.

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